A PLANE SLIGHTLY CURVED JET OF A PERFECT INCOMPRESSIBLE FLUID

(PLOSKAIA SLABOISKRIVLENNAIA STRUIA IDEAL'NOI NESZHIMAEMOI ZHIDKOSTI)

PMM Vol.28, № 3, 1964, pp.564-566

A.P.FROLOV
(Moscow)

(Received November 2, 1963)

Steady potential flow of fluid in thin layers bounded by curvilinear surfaces was investigated in detail in [1 and 2] for the conditions that the vector of flow velocity does not vary along lines normal to the surfaces defining the layer, and that the velocity components along the normal are equal to zero.

Frankl' [3] who studied potential jet flows of fluid on the surface of a solid body, obtained an approximate equation for the potential of velocities.

A solution in closed form is found below for the problem of plane-parallel jet flow on a curvilinear surface of a small curvature.

1. Let us examine a jet of a perfect incompressible fluid bounded by a slightly curved wall L_1 and a region of constant pressure $p = p_0$ (Fig.1). An auxiliary orthogonal curvilinear coordinate system is selected such that the axis s is oriented along the free streamline L_0. In this curvilinear system Euler's equations and continuity equations have the following form [4]

$$\frac{v_s}{H_s}\frac{\partial v_s}{\partial s}+\frac{v_r}{H_r}\frac{\partial v_s}{\partial r}+\frac{v_s v_r}{H_s H_r}\frac{\partial H_s}{\partial r}-\frac{v_r^2}{H_r H_s}\frac{\partial H_r}{\partial s}=-\frac{1}{\rho H_s}\frac{\partial p}{\partial s}$$

$$\frac{v_r}{H_r}\frac{\partial v_r}{\partial r}+\frac{v_s}{H_s}\frac{\partial v_r}{\partial s}+\frac{v_r v_s}{H_r H_s}\frac{\partial H_r}{\partial s}-\frac{v_s^2}{H_s H_r}\frac{\partial H_s}{\partial r}=-\frac{1}{\rho H_r}\frac{\partial p}{\partial r} \qquad (1.1)$$

$$\frac{1}{H_s}\frac{\partial v_s}{\partial s}+\frac{1}{H_r}\frac{\partial v_r}{\partial r}+\frac{1}{H_s H_r}\left(v_s\frac{\partial H_r}{\partial s}+v_r\frac{\partial H_s}{\partial r}\right)=0 \qquad \left(H_r=1,\ H_s=1+\frac{r}{R(s)}\right)$$

where H_r and H_s are Lamé parameters, $R(s)$ is the radius of curvature of the jet. It is assumed that the thickness of the jet $h(s) \ll R(s)$, and terms of the order $h(s)/R(s)$. are omitted in Equations (1.1). Then equations of the following form are obtained:

$$v_s\frac{\partial v_s}{\partial s}+v_r\frac{\partial v_s}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial s}$$

$$-K(s)\,v_s^2=-\frac{1}{\rho}\frac{\partial p}{\partial r}, \qquad \frac{\partial v_s}{\partial s}+\frac{\partial v_r}{\partial r}=0 \qquad (1.2)$$

Here $K(s)$ is the curvature of the jet.

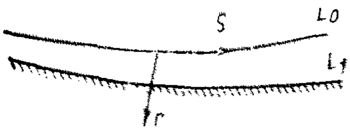

Fig. 1

It is assumed that the function $K(s)$ is continuously differentiable.

Using equations of continuity a stream function $\psi(r,s)$ is introduced, and the pressure $p(r,s)$ is eliminated from Equations (1.2). For determination of the stress function a partial nonlinear equation of third order is obtained

$$\frac{\partial\psi}{\partial r}\frac{\partial^3\psi}{\partial s\partial r^2}-\frac{\partial\psi}{\partial s}\frac{\partial^3\psi}{\partial r^3}+\frac{dK}{ds}\left(\frac{\partial\psi}{\partial r}\right)^2+2K(s)\frac{\partial\psi}{\partial r}\frac{\partial^2\psi}{\partial r\partial s}=0 \tag{1.3}$$

We will look for a solution of this equation in the form

$$\psi(r,s)=\frac{\eta+f(\eta)}{K(s)} \tag{1.4}$$

where the nondimensional quantity $\eta = rK(s)$. Since L_0 is a streamline, we can set the following for $r = 0$

$$\psi(0,s)=0, \qquad f(0)=0 \tag{1.5}$$

Substituting Expression (1.4) for the stream function in Equation (1.3), an ordinary differential equation of third order is obtained for the determination of $f(\eta)$

$$(\eta+f)f'''+(1+f')f''+2\eta(1+f')f''+(1+f')^2=0 \tag{1.6}$$

This equation can be written in the form

$$\frac{d}{d\eta}[(\eta+f)f''+\eta(1+f')^2]=0$$

From this it follows directly that

$$(\eta+f)f''+\eta(1+f')^2=C_1 \tag{1.7}$$

From condition (1.5) it follows that $C_1 = 0$. In this case Equation (1.7) is readily integrated, and for the function $f(\eta)$ the following expression is obtained

$$f(\eta)=\frac{c\eta}{\eta+2}-\eta \qquad (C=\text{const})$$

Consequently, the desired stream function has the form

$$\psi(r,s)=\frac{Cr}{rK(s)+2} \tag{1.8}$$

For the determination of the constant C, the velocity components are found

$$v_s=\frac{\partial\psi}{\partial r}=\frac{2C}{[rK(s)+2]^2}, \qquad v_r=-\frac{\partial\psi}{\partial s}=\frac{Cr^2K'(s)}{[rK(s)+2]^2} \tag{1.9}$$

For $r = 0$ the velocity $v_r = 0$ and $v_s = v_0$, where the velocity v_0 is a constant along the free streamline L_0 and is determined for the known pressure p_0 from the Bernoulli equation. From (1.9) it follows that the constant $C = 2v_0$.

Together with the streamline L_0, the curvilinear wall L_1, for which the equation in the selected system of coordinates is $r=h(s)$, is also a streamline, i.e. $\psi[h(s),s] = Q$, where Q is the amount of fluid in the jet. Utilizing (1.8) we find from this relationship

$$h(s)=\frac{2Q}{2v_0-QK(s)} \tag{1.10}$$

In this fashion the flow is completely determined when the curvature of the stream is known. Thus, for velocity $v_1(s)$ along the wall L_1 the following expression is obtained from (1.9) for $r = h(s)$

$$v_1^2(s)=\frac{4v_0^2[4+h^4(s)K'^2(s)]}{[h(s)K(s)+2]^4}$$

or, substituting $h(s)$ according to (1.10),

$$v_1^2(s) = \frac{4v_0^2}{(2v_0 / Q)^4}\left[K'^2(s) + \frac{1}{4}\left(K - \frac{2v_0}{Q}\right)^4\right] \tag{1.11}$$

It is noted that the relationship (1.11) can be regarded as an ordinary differential equation for a curvature of the jet $K(s)$, if the distribution of velocities along the wall L_1 is considered as known.

2. Results obtained are utilized for the solution of the problem of jet impingement on the surface of a heavy fluid (Fig. 2). At the boundary of contact between the jet and the heavy fluid at rest there will be a contact discontinuity of velocities. The pressure which does not undergo a discontinuity along this boundary is taken to be equal to the hydrostatic pressure $p_1 = p_0 - \gamma y(s)$. The phenomenon mentioned is particularly clearly observed on impact of a gaseous jet on the surface of a heavy fluid. Here $x(s)$ and $y(s)$ are equations for the boundary L_1 in Cartesian coordinates (Fig. 2), γ is the specific gravity of the heavy fluid.

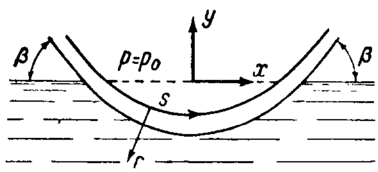

Fig. 2

The velocity $v_1(s)$ along the boundary is determined from Bernoulli's equation

$$v_1^2(s) = v_0^2 + 2\gamma y(s) \tag{2.1}$$

Subsequently the curvature of the jet is determined from (1.11).

The derivative dh/ds is a small quantity of the order $h(s)/R(s)$, and therefore in the exact condition $v_r = v_1 \sin\theta$ we can set $\sin\theta \approx \tan\theta = dh/ds$ (where θ is the angle of inclination of velocity to the s-axis).

Therefore, when a small value for the ratio $h(s)/R(s)$ is specified, using the approximate relationship

$$v_r = v_1 \frac{dh}{ds} \tag{2.2}$$

instead of Equation (1.11), we obtain

$$\frac{4v_0}{[h(s)K(s) + 2]^2} = v_1(s), \quad h(s) = \frac{Q}{\sqrt{v_0 v_1(s)}} \tag{2.3}$$

and according to (1.10) the curvature of the jet is

$$K(s) = \frac{2v_0}{Q}\left[1 - \left(\frac{v_1(s)}{v_0}\right)^{1/2}\right] \tag{2.4}$$

Substituting the value $v_1(s)$ from (2.1) into (2.4) a relationship of the following form is obtained

$$K(s) = \frac{2}{Q}\left(v_0 - \sqrt{v_0\sqrt{v_0^2 + 2\gamma y(s)}}\right) \tag{2.5}$$

In view of the thinness of the jet, with accuracy to small quantities of second order the curvature of the jet is taken to be equal to the curvature of the line L_1. Then, if the curvature of the line L_1 is assumed to be known from (2.5), the equations for the curve have the form [5]

$$y(s) = \int_0^s \sin\left[\int_0^s K(s)\,ds + \alpha_0\right] ds + y(0) \tag{2.6}$$

$$(\alpha_0 = \text{const})$$

$$x(s) = \int_0^s \cos\left[\int_0^s K(s)\,ds + \alpha_0\right] ds + x(0) \tag{2.7}$$

Eliminating the curvature $K(s)$ from (2.5) and (2.6), we obtain

$$\frac{d^2y\,/\,ds^2}{\sqrt{1-(dy\,/\,ds)^2}}=\frac{2}{Q}\left(v_0-\sqrt{v_0\sqrt{v_0^2+2\gamma y\,(s)}}\right) \tag{2.8}$$

This equation is solved in an elementary fashion by substituting

$$dy\,/\,ds=B\,(y).$$

Finally, the problem of determining the shape of jet reduces to the quadrature

$$s=\pm\int\left\{1-\left[\frac{2}{Q}\left(v_0y-\frac{2\sqrt{v_0}}{5\gamma}\,(v_0^2+2\gamma y)^{5/4}\right)+C_2\right)\right]^2\right\}^{-1/2}dy \tag{2.9}$$

The curvature $K(s)$ is determined from the known function $y(s)$, then $x(s)$ is found from (2.7). However, by making use of the relationship $dx^2+dy^2=ds^2$ the parameter s might be eliminated

$$\frac{dy}{dx}=\pm\frac{\{1-[2Q^{-1}(v_0y-{}^2/_5v_0^{1/2}\gamma^{-1}(v_0^2+2\gamma y)^{5/4})+C_2]^2\}^{1/2}}{\{2Q^{-1}[v_0y-{}^2/_5v_0^{1/2}\gamma^{-1}(v_0^2+2\gamma y)^{5/4}]+C_2\}^{-1}} \tag{2.10}$$

The constant C_2 is determined from the known angle of incidence of the jet. If the angle of incidence for the jet is set equal to β (Fig. 2) we obtain

$$C_2=\frac{4v_0^3}{5\gamma Q}+\cos\beta$$

From Equation (2.10) we can determine directly the maximum penetration depth of the jet into the fluid (setting $dy/dx=0$), and also the angle of exit for the jet It turns out that the angle of exit is equal to the angle of incidence for the jet.

BIBLIOGRAPHY

1. Golubeva, O.V., Issledovanie dvizhenii zhidkosti po krivolineinym poverkhnostiam (Investigation of fluid flow on curvilinear surfaces). Uch.zap. MOPI Vol.18, № 2, 1951.
2. Golubeva, O.V., Uravnenie ustanovivshikhsia potentsial'nykh dvizhenii ideal'noi neszhimaemoi zhidkosti v plenke peremennoi tolshchiny, raspolozhennoi na krivolineinoi poverkhnosti (Equations for steady potential flows of a perfect incompressible fluid in a film of varying thickness located on a curvilinear surface). Uch.zap.MOPI Vol.33, 1955.
3. Frankl', F.I., Priblizhennyi raschet struinogo potentsial'nogo techeniia zhidkosti, rasprostraniaiushchegosia po poverkhnosti tverdogo tela tonkim sloem (Approximate calculation of potential jet flow of a fluid propagating in a thin layer over the surface of a solid body). *PMM* Vol.24, № 2, 1960.
4. Kochin, N.E., Kibel', I.A. and Roze, N.V., Teoreticheskaia gidromekhanika (Theoretical Hydromechanics), part 1 and 2, Fizmatgiz, 1963.
5. Rashevskii, P.K., Kurs differentsial'noi geometrii (A Course in Differential Geometry). Gostekhteoretizdat, 1950.

Translated by B.D.